

The Lorentz Group, Noncommutative Space-Time, and Nonlinear Electrodynamics in Majorana-Oppenheimer Formalism

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Abstract

Non-linear electrodynamics arising in the frames of field theories in non-commutative space-time is examined on the base of the Riemann-Silberstein-Majorana-Oppenheimer formalism. The problem of form-invariance of the non-linear constitutive relations governed by six non-commutative parameters $\theta_{kl} \sim \mathbf{K} = \mathbf{n} + i\mathbf{m}$ is explored in detail on the base of the complex orthogonal group theory $SO(3, \mathbb{C})$. Two Abelian 2-parametric small groups, isomorphic to each other in abstract sense, and leaving unchangeable the extended constitutive relations at arbitrary six parameters θ_{kl} of effective media have been found, their realization depends explicitly on invariant length \mathbf{K}^2 . In the case of non-vanishing length a special reference frame in which the small group has the structure $SO(2) \otimes SO(1, 1)$ has been found. In isotropic case no such reference frame exists. The way to interpret both Abelian small groups in physical terms consists in factorizing corresponding Lorentz transformations into Euclidean rotations and boosts. In the context of general study of various dual symmetries in non-commutative field theory, it is demonstrated explicitly that the non-linear constitutive equations in non-commutative electrodynamics are not invariant under continuous dual rotations, instead only invariance under discrete dual transformation exists.

1 Introduction

As known [1-15] interest in field theory models in a non-commutative space-time has been grown notably after creating in [15] a general algorithm to relate usual Yang-Mills gauge models to their non-commutative counterparts. There appears a great deal of new physical problems to investigate, besides the question of the hypothetic coordinate non-commutativity has become of practically testable nature. Noticeable progress in describing symmetry of non-commutative spaces was achieved on the base of twisted Poincare group.

For instance, the mapping by Seiberg – Witten refers the non-commutative extension of electrodynamics to the usual microscopic Maxwell theory with special non-linear constitutive relations. Examining all possible symmetries of these new constitutive relations seems to be a significant point in order to discern the effects of the space-time non-commutativity in observable electromagnetic non-linear effects.

The problem of form-invariance of the non-commutativity structural equations (see below) was considered in the literature. Several simple non-commutative parameters were listed which

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allow for existence of some residual Lorentz symmetry – the later is recognized to have the structure $SO(2) \otimes SO(1, 1)$.

The aim of the present article is to establish subgroups of the Lorentz group leaving form-invariant the commutator of space-time coordinates with arbitrary noncommutative antisymmetric matrix. The starting commutative relationship transform with respect to Lorentz group according to

$$[L_k^a x_a, L_l^b x_b]_- = i L_k^a L_l^b \theta_{ab} = i \theta'_{kl}. \quad (1)$$

There exist several different views on the transforms of the matrix $\theta^{\mu\nu}$. Evidently, we aim at extension of Lorentz invariant models in ordinary Minkowski space-time to models in non-commutative space-time.

We might consider skew-symmetric object $\theta^{\mu\nu}$ just as a tensor under the Lorentz group, without any physically preferable reference frame. Therefore, six parameters involved in $\theta^{\mu\nu}$ -entity depend on the choice of the reference frame, they behave like all other tensor or spinor objects in physics. Within that approach any field model in non-commutative space-time must involve only Lorentz covariant constructs. Similar line of argument was used by Herman Minkowski when creating microscopic electrodynamics in moving medium. As known, according to Minkowski constitutive equations, Euclidean rotations do not change parameters of the uniform medium, ϵ and μ , whereas all boost transform them into new ones depending on the velocity vector \vec{V} of the reference frame. Differently, it sounds as follows: small Lorentz group leaving invariant parameters of an uniform media coincides with real orthogonal group $SO(3.R)$. Below we consider a similar problem in the frames of a non-commutative electrodynamics.

The most radical attitude to the transforms of $\theta^{\mu\nu}$ -entity may be formulated as follows: six parameters involved in $\theta^{\mu\nu}$ -entity provide us with new six fundamental constants. However, immediately one questions may be posed: in with reference frame we must take these fundamental constants. And then what are symmetry transformations (small Lorentz group) leaving invariant these six parameters. In a sense, in this point we turn back to the old question on existence of a fundamental ether. No solution for ether problem has found till now, so it is hardly reasonable to reanimate the old unsolved puzzle in new embodiment.

Evidently, presented below simple mathematical treatment is of value in any case, irrespective of the choice between two mentioned views. As mentioned, several particular examples of such small (or stability) subgroups were noticed in the literature, so our analysis extends and completes previous considerations. In a sense, the problem may be straightforwardly solved with the help of old and well elaborated technique in the theory of the Lorentz group [12], [5]. A basic tool used in this article is the theory of complex rotation group $SO(3.C)$, isomorphic to the Lorentz group, and the theory of the special linear group $SL(2.C)$, spinor covering for Lorentz group. So to deal with the non-linear Maxwell theory we employ the known Riemann-Silberstein-Majorana-Oppenheimer approach – for more detail and references see [6].

In the context of general study of various dual symmetries in non-commutative field theory one other problem will be considered: it is demonstrated explicitly that the known non-linear constitutive equations arising from non-commutative electrodynamics in the first order approximation are not invariant under continuous dual rotations, instead only invariance under discrete dual transformation exists.

2 Basic facts in the Lorentz group, notation

Let us recall basic facts in the theory of the Lorentz group and related to it, focusing on its parametrization [12], [5]. Let us start with the real rotation group $SO(3.R)$ and its covering $SU(2)$:

$$\begin{aligned} B(n) &= n_0 - i \mathbf{n} \vec{\sigma} = \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \mathbf{e} \vec{\sigma}, \quad n_0^2 + \mathbf{n}^2 = 1, \\ O(n) &= I + 2 [n_0 \mathbf{n}^\times + (\mathbf{n}^\times)^2], \quad (\mathbf{n}^\times)_{il} = -\epsilon_{ilj} n_j, \\ O(n) &= \begin{vmatrix} 1 - 2(n_2^2 + n_3^2) & -2n_0n_3 + 2n_1n_2 & +2n_0n_2 + 2n_1n_3 \\ +2n_0n_3 + 2n_1n_2 & 1 - 2(n_3^2 + n_1^2) & -2n_0n_1 + 2n_2n_3 \\ -2n_0n_2 + 2n_1n_3 & +2n_0n_1 + 2n_2n_3 & 1 - 2(n_1^2 + n_2^2) \end{vmatrix}. \end{aligned} \quad (2)$$

The composition rule in the unitary group is

$$n''_0 = n'_0 n_0 - \mathbf{n}' \mathbf{n}, \quad \mathbf{n}'' = n'_0 \mathbf{n} + n_0 \mathbf{n}' + \mathbf{n}' \times \mathbf{n}; \quad (3)$$

transition to explicit parametrization of the rotation group is achieved by the introduction of the Gibbs' 3-vector (for more details see in [12]):

$$\begin{aligned} \mathbf{c} &= \frac{\mathbf{n}}{n_0} = \text{tg} \frac{\alpha}{2} \mathbf{e}, \quad \mathbf{c}'' = \frac{\mathbf{c}' + \mathbf{c} + \mathbf{c}' \times \mathbf{c}}{1 - \mathbf{c}' \mathbf{c}}, \quad O(\mathbf{c}) = I + 2 \frac{\mathbf{c}^\times + (\mathbf{c}^\times)^2}{1 + \mathbf{c}^2} \\ &= \frac{1}{1 + \mathbf{c}^2} \begin{vmatrix} 1 + \mathbf{c}^2 - 2(c_2^2 + c_3^2) & -2c_3 + 2c_1c_2 & +2c_2 + 2c_1c_3 \\ +2c_3 + 2c_1c_2 & 1 + \mathbf{c}^2 - 2(c_3^2 + c_1^2) & -2c_1 + 2c_2c_3 \\ -2c_2 + 2c_1c_3 & +2c_1 + 2c_2c_3 & 1 + \mathbf{c}^2 - 2(c_1^2 + c_2^2) \end{vmatrix}. \end{aligned}$$

One should note the peculiarity: if $n_0 = 0$ (when $\alpha = \pi$), then

$$B(n) = -i \mathbf{n} \vec{\sigma}, \quad \mathbf{c} = \infty \mathbf{e}, \quad O(\infty \mathbf{e}) = I + 2 (\mathbf{e}^\times)^2.$$

Rotation matrices (2) can be written differently through (α, \mathbf{e}) .

$$O(\alpha, \mathbf{e}) = \begin{vmatrix} 1 - F(e_2^2 + e_3^2) & -\sin \alpha e_3 + F e_1 e_2 & \sin \alpha e_2 + F e_1 e_3 \\ \sin \alpha e_3 + F e_1 e_2 & 1 - F(e_3^2 + e_1^2) & -\sin \alpha e_1 + F e_2 e_3 \\ -\sin \alpha e_2 + F e_1 e_3 & \sin \alpha e_1 + F e_2 e_3 & 1 - F(e_1^2 + e_2^2) \end{vmatrix}, \quad (4)$$

where $F = (1 - \cos \alpha)$; at $\alpha = \pi$ it reads

$$O = \begin{vmatrix} 1 - 2(e_2^2 + e_3^2) & +2e_1e_2 & 2e_1e_3 \\ +2e_1e_2 & 1 - 2(e_3^2 + e_1^2) & +2e_2e_3 \\ +2e_1e_3 & 2e_2e_3 & 1 - 2(e_1^2 + e_2^2) \end{vmatrix} = I + 2 (\mathbf{e}^\times)^2.$$

Extension to the special linear group $SL(2.C)$, spinor covering for the (proper orthochronous) Lorentz group L_+^\uparrow , is achieved by formal change $(n_0, -i\mathbf{n})$ to any complex (k_0, \mathbf{k}) :

$$\begin{aligned} B(k_0, \mathbf{k}) &= k_0 + k_j \sigma_j = (n_0 + im_0) + (-in_j + m_j) \sigma_j, \\ \det B &= k_0^2 - \mathbf{k}^2 = n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 + 2i(n_0m_0 + \mathbf{nm}) = 1 \end{aligned} \quad (5)$$

with the following composition rule

$$k''_0 = k'_0 k_0 - \mathbf{k}' \mathbf{k}, \quad \mathbf{k}'' = k'_0 \mathbf{k} + k_0 \mathbf{k}' + i \mathbf{k}' \times \mathbf{k}$$

which coincides with (3) when restricting to subgroup $SU(2)$.

The complex orthogonal group $SO(3.C)$ may be defined as $2 \rightarrow 1$ mapping from $SL(2.C)$, its elements are

$$O(k) = I + 2 [k_0 \mathbf{k}^\times + (\mathbf{k}^\times)^2]$$

$$= \begin{vmatrix} 1 + 2(k_2^2 + k_3^2) & -2ik_0k_3 - 2k_1k_2 & +2ik_0k_2 - 2k_1k_3 \\ +2ik_0k_3 - 2k_1k_2 & 1 + 2(k_3^2 + k_1^2) & -2ik_0k_1 - 2k_2k_3 \\ -2ik_0k_2 - 2k_1k_3 & +2ik_0k_1 - 2k_2k_3 & 1 + 2(k_1^2 + k_2^2) \end{vmatrix}. \quad (6)$$

Euclidean rotations are specified by

$$k_0 = n_0, \quad k_j = -in_j, \quad n_0 = \cos \frac{\alpha}{2}, \quad \mathbf{n} = \sin \frac{\alpha}{2} \mathbf{e};$$

note identities

$$[O(n)]^* = O(n), \quad [O(n)]^{-1} = O(\bar{n}) = [O(n)]^{tr}.$$

Lorentz boosts are specified by

$$k_0 = n_0, \quad \mathbf{k} = \mathbf{m}, \quad n_0 = \text{ch} \frac{\beta}{2}, \quad \mathbf{m} = \text{sh} \frac{\beta}{2} \mathbf{e},$$

$$O = \begin{vmatrix} 1 - (1 - \text{ch} \beta)(e_2^2 + e_3^2) & -i \text{sh} \beta e_3 + G e_1 e_2 & i \text{sh} \beta e_2 + G e_1 e_3 \\ i \text{sh} \beta e_3 + (1 - \text{ch} \beta) e_1 e_2 & 1 - G(e_3^2 + e_1^2) & -i \text{sh} \beta e_1 + G e_2 n_3 \\ -i \text{sh} \beta e_2 + G e_1 e_3 & i \text{sh} \beta e_1 + G e_2 e_3 & 1 - G(e_1^2 + e_2^2) \end{vmatrix}$$

where $G = (1 - \text{ch} \beta)$; note identities

$$[O(n_0, \mathbf{m})]^* = [O(n_0, \mathbf{m})]^{-1} = O(n_0, -\mathbf{m}) = [O(n_0, \mathbf{m})]^{tr}.$$

Let us write down the real Lorentz transformation over 4-vectors:

$$L = \begin{vmatrix} k_0 k_0^* & (-k_0^* k_1 - k_0 k_1^*) & -k_0^* k_2 - k_0 k_2^* & -k_0^* k_3 - k_0 k_3^* \\ -k_0^* k_1 - k_0 k_1^* & k_0 k_0^* & -ik_0^* k_3 + ik_0 k_3^* & +ik_0^* k_2 - ik_0 k_2^* \\ -k_0^* k_2 - k_0 k_2^* & ik_0^* k_3 - ik_0 k_3^* & k_0 k_0^* & -ik_0^* k_1 + ik_0 k_1^* \\ -k_0^* k_3 - k_0 k_3^* & -ik_0^* k_2 + ik_0 k_2^* & +ik_0^* k_1 - ik_0 k_1^* & k_0 k_0^* \end{vmatrix}$$

$$+ \begin{vmatrix} D_0 & i(+k_2 k_3^* - k_3 k_2^*) & i(-k_1 k_3^* + k_3 k_1^*) & i(k_1 k_2^* - k_2 k_1^*) \\ -i(+k_2 k_3^* - k_3 k_2^*) & D_1 & k_1 k_2^* + k_2 k_1^* & k_1 k_3^* + k_3 k_1^* \\ -i(-k_1 k_3^* + k_3 k_1^*) & k_1 k_2^* + k_2 k_1^* & D_2 & k_2 k_3^* + k_3 k_2^* \\ -i(+k_1 k_2^* - k_2 k_1^*) & +k_1 k_3^* + k_3 k_1^* & +k_2 k_3^* + k_3 k_2^* & D_3 \end{vmatrix}$$

where

$$D_0 = k_j k_j^*, \quad D_1 = k_1 k_1^* - k_2 k_2^* - k_3 k_3^*,$$

$$D_2 = k_2 k_2^* - k_1 k_1^* - k_3 k_3^*, \quad D_3 = k_3 k_3^* - k_1 k_1^* - k_2 k_2^*$$

or taking into account $k_a = -in_a + m_a$:

$$L = 2 \begin{vmatrix} (n_0^2 + m_0^2)/2 & n_1 m_0 - n_0 m_1 & n_2 m_0 - n_0 m_2 & n_3 m_0 - n_0 m_3 \\ n_1 m_0 - n_0 m_1 & (n_0^2 + m_0^2)/2 & -n_0 n_3 - m_0 m_3 & n_0 n_2 + m_0 m_2 \\ n_2 m_0 - n_0 m_2 & n_0 n_3 + m_0 m_3 & (n_0^2 + m_0^2)/2 & -n_0 n_1 - m_0 m_1 \\ n_3 m_0 - n_0 m_3 & -n_0 n_2 - m_0 m_2 & 2n_0 n_1 + m_0 m_1 & (n_0^2 + m_0^2)/2 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} D_0/2 & n_2 m_3 - n_3 m_2 & n_3 m_1 - n_1 m_3 & n_1 m_2 - n_2 m_1 \\ -n_2 m_3 + n_3 m_2 & D_1/2 & n_1 n_2 + m_1 m_2 & n_1 n_3 + m_1 m_3 \\ -n_3 m_1 + n_1 m_3 & n_1 n_2 + m_1 m_2 & D_2/2 & n_2 n_3 + m_2 m_3 \\ -n_1 m_2 + n_2 m_1 & n_1 n_3 + m_1 m_3 & n_2 n_3 + m_2 m_3 & D_3/2 \end{vmatrix}$$

where

$$\begin{aligned} D_0 &= n_1^2 + m_1^2 + n_2^2 + m_2^2 + n_3^2 + m_3^2, \\ D_1 &= n_1^2 + m_1^2 - n_2^2 - m_2^2 - n_3^2 - m_3^2, \\ D_2 &= -n_1^2 - m_1^2 + n_2^2 + m_2^2 - n_3^2 - m_3^2, \\ D_3 &= -n_1^2 - m_1^2 - n_2^2 - m_2^2 + n_3^2 + m_3^2. \end{aligned}$$

Let us verify these formulas for Lorentz boosts: $n_0 = \text{ch} \frac{\beta}{2}$, $\mathbf{m} = \text{sh} \frac{\beta}{2} \mathbf{e}$, $\mathbf{e}^2 = 1$; the matrix L reads

$$L = \begin{vmatrix} \text{ch } \beta & -\text{sh } \beta e_1 & -\text{sh } \beta e_2 & -\text{sh } \beta e_3 \\ -\text{sh } \beta e_1 & 1 + (\text{ch } \beta - 1)e_1^2 & (\text{ch } \beta - 1)e_1 e_2 & (\text{ch } \beta - 1)e_1 e_3 \\ -\text{sh } \beta e_2 & (\text{ch } \beta - 1)e_1 e_2 & 1 + (\text{ch } \beta - 1)e_2^2 & (\text{ch } \beta - 1)e_2 e_3 \\ -\text{sh } \beta e_3 & (\text{ch } \beta - 1)e_1 e_3 & (\text{ch } \beta - 1)e_2 e_3 & 1 + (\text{ch } \beta - 1)e_3^2 \end{vmatrix}, \quad (7)$$

which in terms of space-time transformation coincides with the standard form

$$t' = \text{ch } \beta t - \text{sh } \beta (\mathbf{e} \cdot \mathbf{x}), \quad \mathbf{x}' = -\text{sh } \beta \mathbf{e} t + [\mathbf{x} + (\text{ch } \beta - 1) \mathbf{e} (\mathbf{e} \cdot \mathbf{x})].$$

3 The problem of a small group in $SO(3.C)$, non-isotropic case

Let return to eq. (5) and note that the whole set of element of $SL(2.C)$ can be divided into two subsets depending on \mathbf{k} with vanishing or not length. In this Section we consider the non-isotropic case, $\mathbf{k}^2 \neq 0$. Here, one may introduce a (γ, Δ) -parametrization of that subset as follows:

$$\begin{aligned} B(k) &= \sin \frac{\gamma}{2} - i \sin \frac{\gamma}{2} \Delta, \quad \gamma = \alpha + i\beta, \\ \Delta &= \mathbf{N} + i\mathbf{M}, \quad \Delta^2 = (\mathbf{N}^2 - \mathbf{M}^2) + 2i\mathbf{N}\mathbf{M} = 1. \end{aligned} \quad (8)$$

Now, Euclidean rotation and Lorentzian boost are specified respectively by conditions:

$$\beta = 0, \quad \mathbf{M} = 0, \quad \text{and} \quad \alpha = 0, \quad \mathbf{M} = 0.$$

One can express the Lorentz matrix L in terms of the variable (γ, Δ)

$$\gamma = \alpha + i\beta, \quad \Delta = \mathbf{N} + i\mathbf{M};$$

it suffices to allow for the identities

$$\begin{aligned} n_0 &= \cos \frac{\alpha}{2} \text{ch} \frac{\beta}{2}, \quad m_0 = -\sin \frac{\alpha}{2} \text{sh} \frac{\beta}{2} \\ \mathbf{n} &= \sin \frac{\alpha}{2} \text{ch} \frac{\beta}{2} \mathbf{N} - \cos \frac{\alpha}{2} \text{sh} \frac{\beta}{2} \mathbf{M} \\ \mathbf{m} &= \cos \frac{\alpha}{2} \text{sh} \frac{\beta}{2} \mathbf{N} + \sin \frac{\alpha}{2} \text{ch} \frac{\beta}{2} \mathbf{M}. \end{aligned} \quad (9)$$

The set of spinor matrices (8) at any fixed vector Δ , $\Delta^2 = 1$ consists of a 2-parametric subgroup with Abelian group multiplication law:

$$\frac{\gamma''}{2} = \frac{\gamma'}{2} + \frac{\gamma}{2}. \quad (10)$$

Complex rotation matrices O in (γ, Δ) -parametrization look

$$O = \begin{vmatrix} 1 - F(\Delta_2^2 + \Delta_3^2) & -2 \sin \gamma \Delta_3 + F \Delta_1 \Delta_2 & +2 \sin \gamma \Delta_2 + F \Delta_1 \Delta_3 \\ +2 \sin \gamma \Delta_3 + F \Delta_1 \Delta_2 & 1 - F(\Delta_3^2 + \Delta_1^2) & -2 \sin \gamma \Delta_1 + F \Delta_2 \Delta_3 \\ -2 \sin \gamma \Delta_2 + F \Delta_1 \Delta_3 & +2 \sin \gamma \Delta_1 + F \Delta_2 \Delta_3 & 1 - F(\Delta_1^2 + \Delta_2^2) \end{vmatrix}$$

where $F = 1 - \cos \gamma$. We need one simple property of these 2-parametric subgroups (10) an any fixed Δ – each of them leaves invariant a definite complex non-isotropic 3-vector, fixed up to any non-zero complex factor λ (equation (11) is verified by direct calculation)

$$\Delta^2 = 1, \quad O(\gamma, \Delta) \lambda \Delta = \lambda \Delta; \quad (11)$$

$$O_{1j} \Delta_j = [\Delta_1 - F(\Delta_2^2 + \Delta_3^2) \Delta_1 - \sin \gamma \Delta_3 \Delta_2 + \\ + F \Delta_1 \Delta_2^2 + \sin \gamma \Delta_2 \Delta_3 + F \Delta_1 \Delta_3^2] = \Delta_1,$$

$$O_{2j} \Delta_j = [\sin \gamma \Delta_3 \Delta_1 + F \Delta_1^2 \Delta_2 + \Delta_2 - \\ - F(\Delta_3^2 + \Delta_1^2) \Delta_2 - \sin \gamma \Delta_1 \Delta_3 + F \Delta_2 \Delta_3^2] = \Delta_2,$$

$$O_{3j} \Delta_j = [-\sin \gamma \Delta_2 \Delta_1 + F \Delta_1^2 \Delta_3 + \sin \gamma \Delta_1 \Delta_2 + \\ + (1 - \cos \gamma) \Delta_2^2 \Delta_3 + \Delta_3 - (1 - \cos \gamma) (\Delta_1^2 + \Delta_2^2) \Delta_3] = \Delta_3.$$

Given arbitrary non-isotropic complex vector \mathbf{K} , to construct a corresponding small subgroup in $SO(3.C)$, it suffices to have found a corresponding vector Δ normalized on $+1$. Let us detail this task:

$$\begin{aligned} \mathbf{K} &= \mathbf{n} + i \mathbf{m} = K \Delta, & \Delta &= \mathbf{N} + i \mathbf{M}, \\ \Delta^2 &= 1, & \mathbf{N}^2 - \mathbf{M}^2 &= 1, & 2i \mathbf{N} \mathbf{M} &= 0, \\ K^2 &= (\mathbf{n}^2 - \mathbf{m}^2) + 2i \mathbf{n} \mathbf{m} = I_1 + i I_2, & I_1, I_2 &= \text{inv}, \end{aligned}$$

that is

$$\mathbf{K} = \mathbf{n} + i \mathbf{m} = K \Delta = \pm \sqrt{(\mathbf{n}^2 - \mathbf{m}^2) + 2i \mathbf{n} \mathbf{m}} (\mathbf{N} + i \mathbf{M}). \quad (12)$$

Complex invariant K^2 may be presented differently

$$\begin{aligned} K^2 &= (\mathbf{n}^2 - \mathbf{m}^2) + 2i \mathbf{n} \mathbf{m} = I (\cos 2\mu + i \sin 2\mu), \\ I &= +\sqrt{(\mathbf{n}^2 - \mathbf{m}^2)^2 + 4(\mathbf{n} \mathbf{m})^2}, \\ \cos 2\mu &= \frac{I_1}{\sqrt{I_1^2 + I_2^2}} = \frac{\mathbf{n}^2 - \mathbf{m}^2}{\sqrt{(\mathbf{n}^2 - \mathbf{m}^2)^2 + 4(\mathbf{n} \mathbf{m})^2}}, \\ \sin 2\mu &= \frac{I_2}{\sqrt{I_1^2 + I_2^2}} = \frac{2\mathbf{n} \mathbf{m}}{\sqrt{(\mathbf{n}^2 - \mathbf{m}^2)^2 + 4(\mathbf{n} \mathbf{m})^2}}. \end{aligned} \quad (13)$$

Therefore, the complex \mathbf{K} may be written in the form

$$\begin{aligned}\mathbf{K} &= \mathbf{n} + i \mathbf{m} = \sqrt{I e^{2i\mu}} (\mathbf{N} + i\mathbf{M}) \\ &= [(\mathbf{n}^2 - \mathbf{m}^2)^2 + 4(\mathbf{n} \mathbf{m})^2]^{1/4} e^{i\mu} (\mathbf{N} + i\mathbf{M}) ;\end{aligned}$$

from whence one obtains an expression for $\mathbf{N} + i\mathbf{M}$:

$$\mathbf{N} + i\mathbf{M} = \frac{e^{-i\mu} (\mathbf{n} + i \mathbf{m})}{[(\mathbf{n}^2 - \mathbf{m}^2)^2 + 4(\mathbf{n} \mathbf{m})^2]^{1/4}} . \quad (14)$$

In two particular cases, these formulas are much simplified:

$$\underline{(I_1 \neq 0, \quad I_2 = 0)}$$

$$\begin{aligned}I(a) \quad \mathbf{n}^2 > \mathbf{m}^2, \quad \mu = 0, \\ \mathbf{n} + i \mathbf{m} = \sqrt{\mathbf{n}^2 - \mathbf{m}^2} \frac{\mathbf{n} + i\mathbf{m}}{\sqrt{\mathbf{n}^2 - \mathbf{m}^2}} \equiv \sqrt{\mathbf{n}^2 - \mathbf{m}^2} (\mathbf{N} + i\mathbf{M}),\end{aligned} \quad (15)$$

$$\begin{aligned}I(b) \quad \mathbf{n}^2 < \mathbf{m}^2, \quad \mu = \frac{\pi}{2}, \\ \mathbf{n} + i \mathbf{m} = \sqrt{\mathbf{n}^2 - \mathbf{m}^2} \frac{(-i)(\mathbf{n} + i\mathbf{m})}{\sqrt{-(\mathbf{n}^2 - \mathbf{m}^2)}} \equiv \sqrt{\mathbf{n}^2 - \mathbf{m}^2} (\mathbf{N} + i\mathbf{M}),\end{aligned} \quad (16)$$

$$\begin{aligned}\underline{(I_1 = 0, \quad I_2 \neq 0)}, \quad \cos 2\mu = 0, \quad \sin 2\mu = \frac{\mathbf{n} \mathbf{m}}{\sqrt{(\mathbf{n} \mathbf{m})^2}}, \\ II(a) \quad \mathbf{n} \mathbf{m} > 0, \quad \mu = \frac{\pi}{4}, \\ \mathbf{n} + i \mathbf{m} = \sqrt{+2(\mathbf{n} \mathbf{m})} \frac{e^{-i\pi/4} (\mathbf{n} + i\mathbf{m})}{\sqrt{+2(\mathbf{n} \mathbf{m})}} \equiv \sqrt{+2(\mathbf{n} \mathbf{m})} (\mathbf{N} + i\mathbf{M}),\end{aligned} \quad (17)$$

$$\begin{aligned}II(b) \quad \mathbf{n} \mathbf{m} > 0, \quad \mu = \frac{\pi}{4}, \\ \mathbf{n} + i \mathbf{m} = \sqrt{2(\mathbf{n} \mathbf{m})} \frac{e^{-i3\pi/4} (\mathbf{n} + i\mathbf{m})}{\sqrt{-2(\mathbf{n} \mathbf{m})}} \equiv \sqrt{2(\mathbf{n} \mathbf{m})} (\mathbf{N} + i\mathbf{M}).\end{aligned} \quad (18)$$

Turning back to the main relationship

$$\begin{aligned}\mathbf{K}^2 \neq 0, \quad \mathbf{K} = K \boldsymbol{\Delta}, \quad \boldsymbol{\Delta}^2 = 1, \\ O(\gamma, \boldsymbol{\Delta}) \quad K \boldsymbol{\Delta} = K \boldsymbol{\Delta}, \quad \boldsymbol{\Delta}^2 = 1,\end{aligned} \quad (19)$$

we note two special cases when the sense of the parameter $\gamma = \alpha + i\beta$ is evident in physical terms:

the first

$$\begin{aligned}
\Delta &= (N + iM) \mathbf{e} , & \gamma &= \alpha , \\
B &= \cos \alpha - i \sin \alpha \mathbf{e} \vec{\sigma} , \\
O(\alpha, \mathbf{e}) (N + iM) \mathbf{e} &= (N + iM) \mathbf{e} , \\
O(\alpha, \mathbf{e}) &\in SO(2) ;
\end{aligned} \tag{20}$$

the second

$$\begin{aligned}
\Delta &= (N + iM) \mathbf{e} , & \gamma &= i \beta , \\
B &= \text{ch } \beta + \text{sh } \beta \mathbf{e} \vec{\sigma} , \\
O(i\beta, \mathbf{e}) (N + iM) \mathbf{e} &= (N + iM) \mathbf{e} , \\
O(i\beta, \mathbf{e}) &\in SO(1, 1) .
\end{aligned} \tag{21}$$

In particular, the above variants $I(a), I(b)$ are of that type:

$$\begin{aligned}
I(a) \quad \mathbf{m} &= 0 , \mathbf{K} = \mathbf{n} = \sqrt{+\mathbf{n}^2} \frac{\mathbf{n}}{\sqrt{+\mathbf{n}^2}} , \\
I(b) \quad \mathbf{n} &= 0 , \mathbf{K} = i \mathbf{m} = \sqrt{-\mathbf{m}^2} \frac{\mathbf{m}}{\sqrt{\mathbf{m}^2}} .
\end{aligned} \tag{22}$$

4 On reduction of a complex non-isotropic vector to a real form

Let us demonstrate that the case of an arbitrary complex vector Δ , $\Delta^2 = 1$ always can be reduced to a real form by means of an appropriate Lorentz transformation. To this end, let start with any complex vector of unit length:

$$\begin{aligned}
\Delta &= \mathbf{N} + i\mathbf{M} , \mathbf{N}^2 - \mathbf{M}^2 = 1 , \mathbf{N} \mathbf{M} = 0 , \\
\mathbf{N} &= \text{ch } \rho \mathbf{N}_0 , \quad \mathbf{N}_0^2 = 1 , \\
\mathbf{M} &= \text{sh } \rho \mathbf{M}_0 , \mathbf{M}_0^2 = 1 , \mathbf{N}_0 \mathbf{M}_0 = 0 ;
\end{aligned} \tag{23}$$

and find a matrix $S \in SO(3, C)$ satisfying equation

$$S(\mathbf{N} + i\mathbf{M}) = \mathbf{e} + i0 , \mathbf{e}^2 = +1 . \tag{24}$$

Eq. (24) can be written differently

$$\frac{S + S^*}{2} (\mathbf{N} + i\mathbf{M}) + \frac{S - S^*}{2} (\mathbf{N} + i\mathbf{M}) = \mathbf{e} + i0.$$

With the notation

$$\frac{S + S^*}{2} = R , \quad \frac{S - S^*}{2} = -i J , \quad S = R - iJ$$

we get two equations

$$R \mathbf{N} + J \mathbf{M} = \mathbf{K}_0 , \quad R \mathbf{M} - J \mathbf{N} = 0 ;$$

they can be written as

$$\begin{aligned} R \operatorname{ch} \rho \mathbf{N}_0 + J \operatorname{sh} \rho \mathbf{M}_0 &= \mathbf{e} , \\ R \operatorname{sh} \rho \mathbf{M}_0 &= +J \operatorname{ch} \rho \mathbf{N}_0 . \end{aligned} \quad (25)$$

Second relation in (25) is equivalent to

$$J^{-1} R \operatorname{th} \rho \mathbf{M}_0 = + \mathbf{N}_0 . \quad (26)$$

However, an orthogonal rotation $O_1 = O(\mathbf{c}_1)$, changing a unite length vector \mathbf{M}_0 into another vector unit length vector \mathbf{N}_0 is well known [12]

$$J^{-1} R \operatorname{th} \rho = O_1 , \quad O_1 \mathbf{M}_0 = \mathbf{N}_0 , \quad \mathbf{c}_1 = \frac{\mathbf{M}_0 \times \mathbf{N}_0}{1 + \mathbf{M}_0 \mathbf{N}_0} . \quad (27)$$

Substituting this into the first equation in (25) we get

$$\operatorname{sh} \rho (R J^{-1} R + J) \mathbf{M}_0 = \mathbf{e} . \quad (28)$$

Rotation transforming the vector \mathbf{M}_0 into \mathbf{e} (note it as O_2) is

$$\operatorname{sh} \rho (R J^{-1} R + J) = O_2 , \quad O_2 \mathbf{M}_0 = \mathbf{e} , \quad \mathbf{c}_2 = \frac{\mathbf{M}_0 \times \mathbf{e}}{1 + \mathbf{M}_0 \mathbf{e}} . \quad (29)$$

Therefore, we know expressions for two matrices O_1 and O_2 – see (27) and (29), in terms of which two other R and J are given:

$$J^{-1} R = \frac{O_1}{\operatorname{th} \rho} , \quad R J^{-1} R + J = \frac{O_2}{\operatorname{sh} \rho} . \quad (30)$$

Solving eqs. (30) is quite elementary:

$$R = \frac{J O_1}{\operatorname{th} \rho} ;$$

and substitution this R into second equation in (30) we get

$$\frac{J O_1}{\operatorname{th} \rho} J^{-1} \frac{J O_1}{\operatorname{th} \rho} + J = \frac{O_2}{\operatorname{sh} \rho} , \quad J (\operatorname{ch}^2 \rho O_1^2 + \operatorname{sh}^2 \rho) = \operatorname{sh} \rho O_2 . \quad (31)$$

Thus J and R have been found:

$$\begin{aligned} J &= \operatorname{sh} \rho O_2 (\operatorname{ch}^2 \rho O_1^2 + \operatorname{sh}^2 \rho)^{-1} , \\ R &= \operatorname{ch} \rho O_2 (\operatorname{ch}^2 \rho O_1^2 + \operatorname{sh}^2 \rho)^{-1} O_1 ; \end{aligned} \quad (32)$$

correspondingly, the the S transformation we need is

$$\begin{aligned} S &= R - iJ \in SO(3.C) , \\ S &= O_2 (\operatorname{ch}^2 \rho O_1^2 + \operatorname{sh}^2 \rho)^{-1} [\operatorname{ch} \rho O_1 - i \operatorname{sh} \rho] . \end{aligned} \quad (33)$$

One may note one special case to choose the vector \mathbf{e} . Indeed, let it be $\mathbf{e} = \mathbf{N}_0$, $O_2 = O_1$ which leads to

$$\begin{aligned} J &= \text{sh } \rho \, O_1 \, (\text{ch}^2 \rho \, O_1^2 + \text{sh}^2 \rho)^{-1} , \\ R &= \text{ch } \rho \, O_1 \, (\text{ch}^2 \rho \, \Pi^2 + \text{sh}^2 \rho)^{-1} \, O_1 . \end{aligned} \quad (34)$$

Besides, one may choose the variant $\mathbf{e} = \mathbf{M}_0$, $O_2 = I$, then we arrive at

$$\begin{aligned} J &= \text{sh } \rho \, (\text{ch}^2 \rho \, O_1^2 + \text{sh}^2 \rho)^{-1} , \\ R &= \text{ch } \rho \, (\text{ch}^2 \rho \, O_1^2 + \text{sh}^2 \rho)^{-1} \, O_1 . \end{aligned} \quad (35)$$

Let us turn again to the stationary subgroup problem:

$$\begin{aligned} \mathbf{K} &= \mathbf{n} + i \, \mathbf{m} = K \, \boldsymbol{\Delta} , \, \boldsymbol{\Delta}^2 = 1 , \\ K &= \sqrt{I_1 + i I_2} , \, I_1 = \mathbf{n}^2 - \mathbf{m}^2 , \, I_2 = 2i \, \mathbf{n} \mathbf{m} \\ O(\gamma, \boldsymbol{\Delta}) \, \sqrt{I_1 + i I_2} \, \boldsymbol{\Delta} &= \sqrt{I_1 + i I_2} \, \boldsymbol{\Delta} , \implies \\ O(\gamma, \mathbf{N} + i \mathbf{N}) \, (\mathbf{n} + i \mathbf{m}) &= (\mathbf{n} + i \mathbf{m}) , \end{aligned} \quad (36)$$

where

$$\mathbf{n} + i \mathbf{m} = \sqrt{I_1 + i I_2} \, (\mathbf{N} + i \mathbf{M}) . \quad (37)$$

Therefore, the main stationary equation in an arbitrary non-isotropic case may be written as

$$O(\gamma, \boldsymbol{\Delta}) = \frac{\mathbf{n} + i \mathbf{m}}{\sqrt{I_1 + i I_2}} (\mathbf{n} + i \mathbf{m}) = (\mathbf{n} + i \mathbf{m}) . \quad (38)$$

In turn, with the help of additional Lorentz transformation S according to (24), one may reduce equation (38) to the form

$$SO(\gamma, \boldsymbol{\Delta}) S^{-1} K \, S \, \boldsymbol{\Delta} = K \, S \boldsymbol{\Delta} = K \, (\mathbf{e} + i \, 0)$$

and further, with the use of the known identity in the theory of the rotation group [12], we arrive at the basic relationship with clear interpretation for γ – see (20)-(21):

$$O(\gamma, \mathbf{e}) \sqrt{I_1 + i I_2} \, \mathbf{e} = \sqrt{I_1 + i I_2} \, \mathbf{e} , \, \mathbf{e}^2 = 1 . \quad (39)$$

In particular, the vector \mathbf{e} may be taken as $\mathbf{e} = \mathbf{N}_0$ or $\mathbf{e} = \mathbf{M}_0$. So, the values of invariants, I_1 and I_2 , govern the possible most simple form for commutative parameters \mathbf{n}' and \mathbf{m}' in different reference frames.

$$\begin{aligned} [x_a, x_b]_- &= i \, \theta_{ab} , \quad \theta_{ab} \sim (\mathbf{n} + i \mathbf{m}) , \\ [x'_a, x'_b]_- &= i \, \theta'_{ab} , \quad \theta'_{ab} \sim (\mathbf{n}' + i \mathbf{m}') = \sqrt{I_1 + i I_2} \, \mathbf{e} . \end{aligned} \quad (40)$$

5 On physical meaning of 2-parametric subgroup $O(\gamma = \alpha + i\beta, \Delta)$ at arbitrary reference frame

To have interpreted the complex parameter $\gamma = \alpha + i\beta$ of the subgroup $O(\gamma = \alpha + i\beta, \Delta)$ at arbitrary reference frame, let us decompose the corresponding spinor elements into product of Euclidean rotation and Lorentz boost:

$$\begin{aligned} & \cos \frac{\alpha + i\beta}{2} - i \sin \frac{\alpha + i\beta}{2} (\mathbf{N} + i\mathbf{M}) \vec{\sigma} \\ &= (\cos \frac{a}{2} - i \sin \frac{a}{2} \mathbf{a} \vec{\sigma}) (\text{ch} \frac{b}{2} + \text{sh} \frac{b}{2} \mathbf{b} \vec{\sigma}) ; \end{aligned} \quad (41)$$

it suffices to solve the problem (it is a spinor variant of the well-known problem of factorization of any Lorentz matrix into rotation and boost – see, for instance, in [12]):

$$\begin{aligned} k_0 + \mathbf{k} \vec{\sigma} &= (a_0 - i\mathbf{a} \vec{\sigma})(b_0 + \mathbf{b} \vec{\sigma}) \\ &= (a_0 b_0 - i \mathbf{a} \mathbf{b}) + (a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b} - i b_0 \mathbf{a}) \vec{\sigma} , \\ k_0^* + \mathbf{k}^* \vec{\sigma} &= (a_0 + i\mathbf{a} \vec{\sigma})(b_0 + \mathbf{b} \vec{\sigma}) \\ &= (a_0 b_0 + i \mathbf{a} \mathbf{b}) + (a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b} + i b_0 \mathbf{a}) \vec{\sigma} ; \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} k_0 &= (a_0 b_0 - i \mathbf{a} \mathbf{b}) , & k_0^* &= (a_0 b_0 + i \mathbf{a} \mathbf{b}) , \\ \mathbf{k} &= (a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b} - i b_0 \mathbf{a}) , \\ \mathbf{k}^* &= (a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b} + i b_0 \mathbf{a}) , \end{aligned}$$

or

$$\begin{aligned} \frac{k_0 + k_0^*}{2} &= a_0 b_0 , & \frac{k_0 - k_0^*}{2i} &= -\mathbf{a} \mathbf{b} , \\ \frac{\mathbf{k} + \mathbf{k}^*}{2} &= a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b} , & \frac{i\mathbf{k} - i\mathbf{k}^*}{2} &= b_0 \mathbf{a} . \end{aligned} \quad (42)$$

With additional restrictions:

$$\begin{aligned} a_0^2 + \mathbf{a}^2 &= +1 , & a_0 &= \pm \sqrt{1 - \mathbf{a}^2} , \\ b_0^2 - \mathbf{b}^2 &= +1 , & b_0 &= +\sqrt{1 + \mathbf{b}^2} \geq +1 \end{aligned} \quad (43)$$

eqs. (42) take the form

$$\begin{aligned} n_0 &= \pm \sqrt{1 - \mathbf{a}^2} \sqrt{1 + \mathbf{b}^2} , & m_0 &= -\mathbf{a} \mathbf{b} , \\ \mathbf{m} &= \pm \sqrt{1 - \mathbf{a}^2} \mathbf{b} + \mathbf{a} \times \mathbf{b} , & \mathbf{n} &= \sqrt{1 + \mathbf{b}^2} \mathbf{a} . \end{aligned}$$

From whence it follows

$$\begin{aligned} n_0 &= \pm \sqrt{1 - \mathbf{a}^2} \sqrt{1 + \mathbf{b}^2} , & m_0 &= -\mathbf{a} \mathbf{b} , \\ \frac{\mathbf{m}}{n_0} &= \frac{\mathbf{b}}{\sqrt{1 + \mathbf{b}^2}} + \frac{\mathbf{a}}{\pm \sqrt{1 - \mathbf{a}^2}} \times \frac{\mathbf{b}}{\sqrt{1 + \mathbf{b}^2}} , \\ & \frac{\mathbf{n}}{n_0} = \frac{\mathbf{a}}{\pm \sqrt{1 - \mathbf{a}^2}} . \end{aligned} \quad (44)$$

With the help of variables \mathbf{A}, \mathbf{B} :

$$\frac{\mathbf{b}}{\sqrt{1+\mathbf{b}^2}} = \mathbf{B}, b_0 = \sqrt{1+\mathbf{b}^2} = \frac{1}{\sqrt{1-\mathbf{B}^2}}, \mathbf{b} = \frac{\mathbf{B}}{\sqrt{1-\mathbf{B}^2}},$$

$$\frac{\mathbf{a}}{\pm\sqrt{1-\mathbf{a}^2}} = \pm\mathbf{A}, a_0 = \pm\sqrt{1-\mathbf{a}^2} = \frac{1}{\pm\sqrt{1+\mathbf{A}^2}}, \mathbf{a} = \frac{\mathbf{A}}{\sqrt{1+\mathbf{A}^2}},$$

we get

$$n_0 = \pm \frac{1}{\sqrt{1+\mathbf{A}^2}} \frac{1}{\sqrt{1-\mathbf{B}^2}}, \quad m_0 = - \frac{\mathbf{A}}{\sqrt{1+\mathbf{A}^2}} \frac{\mathbf{B}}{\sqrt{1-\mathbf{B}^2}},$$

$$\frac{\mathbf{m}}{n_0} = \mathbf{B} + \mathbf{A} \times \mathbf{B}, \quad \frac{\mathbf{n}}{n_0} = \mathbf{A}. \quad (45)$$

The vector \mathbf{B} may be resolved into a linear combination $\mathbf{B} = \nu \mathbf{n} + \mu \mathbf{m} + \sigma \mathbf{n} \times \mathbf{m}$ which must obey

$$\frac{\mathbf{m}}{n_0} = \nu \mathbf{n} + \mu \mathbf{m} + \sigma \mathbf{n} \times \mathbf{m} + \frac{\mathbf{n}}{n_0} \times (\nu \mathbf{n} + \mu \mathbf{m} + \sigma \mathbf{n} \times \mathbf{m})$$

or

$$\frac{\mathbf{m}}{n_0} = \nu \mathbf{n} + \mu \mathbf{m} + \sigma \mathbf{n} \times \mathbf{m} + \frac{\mu}{n_0} \mathbf{n} \times \mathbf{m} + \frac{\sigma}{n_0} (\mathbf{n}\mathbf{m}) \mathbf{n} - \frac{\sigma}{n_0} (\mathbf{n}^2) \mathbf{m}.$$

Therefore, we have the system

$$\nu + \frac{\sigma}{n_0} (\mathbf{n}\mathbf{m}) = 0, \quad \frac{1}{n_0} = \mu - \frac{\sigma (\mathbf{n}^2)}{n_0}, \quad \sigma + \frac{\mu}{n_0} = 0$$

with evident solution

$$\sigma = -\frac{1}{n_0^2 + \mathbf{n}^2}, \quad \mu = \frac{n_0}{n_0^2 + \mathbf{n}^2}, \quad \nu = \frac{(\mathbf{n}\mathbf{m})}{n_0} \frac{1}{n_0^2 + \mathbf{n}^2} = -m_0 \frac{1}{n_0^2 + \mathbf{n}^2}.$$

Thus, the factorization we need is found:

$$k_0 + \mathbf{k} \vec{\sigma} = \left(\frac{n_0 - i\mathbf{n} \vec{\sigma}}{\sqrt{n_0^2 + \mathbf{n}^2}} \right) \left(\frac{1}{\sqrt{1-\mathbf{B}^2}} + \frac{\mathbf{B} \vec{\sigma}}{\sqrt{1-\mathbf{B}^2}} \right),$$

$$\mathbf{B} = \frac{n_0 \mathbf{m} - m_0 \mathbf{n} + \mathbf{m} \times \mathbf{n}}{n_0^2 + \mathbf{n}^2}. \quad (46)$$

The problem of factorization may be solved easily with opposite order:

$$k_0 + \mathbf{k} \vec{\sigma} = (b_0 + \mathbf{b} \vec{\sigma})(a_0 - i\mathbf{a} \vec{\sigma}) = (a_0 b_0 - i\mathbf{a} \mathbf{b}) + (a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b} - i b_0 \mathbf{a}) \vec{\sigma},$$

$$k_0^* + \mathbf{k}^* \vec{\sigma} = (b_0 + \mathbf{b} \vec{\sigma})(a_0 + i\mathbf{a} \vec{\sigma}) = (a_0 b_0 + i\mathbf{a} \mathbf{b}) + (a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b} + i b_0 \mathbf{a}) \vec{\sigma}; \quad (47)$$

it reduces to the system (in comparison with (42) only the sign at the vector product has been changed on opposite)

$$k_0 = (a_0 b_0 - i\mathbf{a} \mathbf{b}), \quad k_0^* = (a_0 b_0 + i\mathbf{a} \mathbf{b}),$$

$$\mathbf{k} = (a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b} - i b_0 \mathbf{a}), \quad \mathbf{k}^* = (a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b} + i b_0 \mathbf{a}),$$

or

$$n_0 = a_0 b_0, \quad m_0 = -\mathbf{a} \cdot \mathbf{b}, \quad \mathbf{m} = a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b}, \quad \mathbf{n} = b_0 \mathbf{a}. \quad (48)$$

Further analysis is the same, the final result is

$$k_0 + \mathbf{k} \cdot \vec{\sigma} = \left(\frac{1}{\sqrt{1 - \mathbf{B}^2}} + \frac{\mathbf{B} \cdot \vec{\sigma}}{\sqrt{1 - \mathbf{B}^2}} \right) \left(\frac{n_0 - i \mathbf{n} \cdot \vec{\sigma}}{\sqrt{n_0^2 + \mathbf{n}^2}} \right),$$

$$\mathbf{B} = \frac{n_0 \mathbf{m} - m_0 \mathbf{n} - \mathbf{m} \times \mathbf{n}}{n_0^2 + \mathbf{n}^2}. \quad (49)$$

The factorizations produced can be translated to parameters $(\gamma/2, \Delta)$:

$$B = k_0 + \mathbf{k} \cdot \vec{\sigma} = \cos \frac{\alpha + i\beta}{2} - i \sin \frac{\alpha + i\beta}{2} \Delta, \quad \Delta = \mathbf{N} + i\mathbf{M}$$

with the help of the formulas (9).

6 The problem of a small group in $SO(3, C)$, isotropic case

Let us consider transformations of the group $SL(2, C)$ with isotropic vector \mathbf{k} :

$$k_0 = \pm 1, \quad B = \pm (I + \mathbf{k} \cdot \vec{\sigma}), \quad \mathbf{k}^2 = 0,$$

$$I_1 = \mathbf{n}^2 - \mathbf{m}^2 = 0, \quad I_2 = +2 \mathbf{n} \cdot \mathbf{m} = 0. \quad (50)$$

Evidently, vectors \mathbf{k} are fixed within arbitrary complex numerical factor $\mathbf{k}' = z \Delta$, $\Delta^2 = 0$, therefore one may construct the following 2-parametric subgroups in $SL(2, C)$; bellow we are interested mainly in corresponding elements in the $SO(3, C)$ group when the factor $\delta = \pm 1$, has no effect:

$$\delta'(I + z' \mathbf{k} \cdot \vec{\sigma}) \delta(I + z \mathbf{k} \cdot \vec{\sigma}) = \delta' \delta [I + (z' + z) \mathbf{k} \cdot \vec{\sigma}] \quad (51)$$

and correspondingly $O(z\mathbf{k}) \mathbf{k} = \mathbf{k}$, $\mathbf{k}^2 = 0$ where

$$O(z\mathbf{k}) = \begin{vmatrix} 1 + 2z^2(k_2^2 + k_3^2) & -2zik_3 - 2z^2k_1\Delta_2 & +2zik_2 - 2z^2k_1\Delta_3 \\ +2zik_3 - 2z^2k_1k_2 & 1 + 2z^2(k_3^2 + k_1^2) & -2zik_1 - 2z^2k_2k_3 \\ -2zik_2 - 2z^2k_1k_3 & +2zik_1 - 2z^2k_2k_3 & 1 + 2z^2(k_1^2 + k_2^2) \end{vmatrix}.$$

Formulas are much simplified in particular cases:

$$\mathbf{k} = (k_1, k_2, 0), \quad k_1^2 + k_2^2 = 0,$$

$$\begin{vmatrix} 1 + 2z^2k_2^2 & -2z^2k_1k_2 & +2zik_2 \\ -2z^2k_1k_2 & 1 + 2z^2k_1^2 & -2izk_1 \\ -2izk_2 & +2izk_1 & 1 + 2z^2(k_1^2 + k_2^2) \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ 0 \end{vmatrix} = \begin{vmatrix} k_1 \\ k_2 \\ 0 \end{vmatrix},$$

$$\mathbf{k} = (0, k_2, k_3), \quad k_1^2 + k_2^2 = 0,$$

$$\begin{vmatrix} 1 + 2z^2(k_2^2 + k_3^2) & -2izk_3 & +2izk_2 \\ +2izk_3 & 1 + 2z^2k_3^2 & -2z^2k_2k_3 \\ -2izk_2 & -2z^2k_2k_3 & 1 + 2z^2k_2^2 \end{vmatrix} \begin{vmatrix} 0 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ k_2 \\ k_3 \end{vmatrix},$$

$$\mathbf{k} = (k_1, 0, k_3), \quad k_1^2 + k_3^2 = 0, \quad \left| \begin{array}{ccc} 1 + 2z^2 k_3^2 & -2izk_3 & -2z^2 k_1 k_3 \\ +2izk_3 & 1 + 2z^2 (k_3^2 + k_1^2) & -2izk_1 \\ -2z^2 k_1 k_3 & +2izk_1 & 1 + 2z^2 k_1^2 \end{array} \right| \left| \begin{array}{c} k_1 \\ 0 \\ k_3 \end{array} \right| = \left| \begin{array}{c} k_1 \\ 0 \\ k_3 \end{array} \right|. \quad (52)$$

To reach some base to interpret the complex parameter z in physical terms, we should use the corresponding 4×4 Lorentz matrices $L(\pm(1, -i\mathbf{n} + \mathbf{m}))$. The $z = \lambda e^{i\sigma}$ - freedom in vector \mathbf{k} is described by relation $\mathbf{k}' = \lambda e^{i\sigma} \mathbf{k}$:

$$\begin{aligned} (-i\mathbf{n}' + \mathbf{m}') &= \lambda (\cos \sigma + i \sin \sigma) (-i\mathbf{n} + \mathbf{m}), \\ \mathbf{n}' &= \lambda (\cos \sigma \mathbf{n} - \sin \sigma \mathbf{m}), \quad \mathbf{m}' = \lambda (\sin \sigma \mathbf{n} + \cos \sigma \mathbf{m}), \\ \mathbf{n}'^2 &= \lambda^2 \mathbf{n}^2, \quad \mathbf{m}'^2 = \lambda^2 \mathbf{m}^2, \\ \mathbf{n}' \mathbf{m}' &= 0, \quad \mathbf{n}' \times \mathbf{m}' = \lambda^2 \mathbf{n} \times \mathbf{m}. \end{aligned} \quad (53)$$

To have additional ground to interpret physically the parameter z , let us factorized spinor matrix $B(\pm(1, -i\mathbf{n} + \mathbf{m}))$ into the product of rotation and boost

$$\begin{aligned} a_0^2 + \mathbf{a}^2 &= 1, \quad b_0^2 - \mathbf{b}^2 = 1, \\ \pm [I + (-i\mathbf{n} + \mathbf{m}) \vec{\sigma}] &= (a_0 - i \mathbf{a} \vec{\sigma}) (b_0 + \mathbf{b} \vec{\sigma}), \\ &= a_0 b_0 + a_0 \mathbf{b} \vec{\sigma} - i b_0 \mathbf{a} \vec{\sigma} - i [\mathbf{a} \mathbf{b} + i (\mathbf{a} \times \mathbf{b}) \vec{\sigma}]. \end{aligned} \quad (54)$$

The problem is reduced to the system

$$\begin{aligned} \mathbf{a} \mathbf{b} &= 0, \quad a_0 b_0 = \pm 1, \\ \pm \mathbf{n} &= b_0 \mathbf{a}, \quad \implies \mathbf{a} = a_0 \mathbf{n}, \\ \pm \mathbf{m} &= a_0 \mathbf{b} + (\mathbf{a} \times \mathbf{b}), \quad \implies b_0 \mathbf{m} = \mathbf{b} + \mathbf{n} \times \mathbf{b}. \end{aligned} \quad (55)$$

One can resolve the vector \mathbf{b} into the linear combination $\mathbf{b} = b_0 [\alpha \mathbf{n} + \beta \mathbf{m} + \gamma (\mathbf{n} \times \mathbf{m})]$, from whence it follows

$$\mathbf{m} = \alpha \mathbf{n} + \beta \mathbf{m} + \gamma (\mathbf{n} \times \mathbf{m}) + \beta \mathbf{n} \times \mathbf{m} - \gamma n^2 \mathbf{m},$$

that is

$$\mathbf{b} = b_0 \frac{(\mathbf{m} - \mathbf{n} \times \mathbf{m})}{1 + n^2}. \quad (56)$$

Thus, the factorization has been found:

$$\begin{aligned} \pm [I - i(\mathbf{n} + i \mathbf{m}) \vec{\sigma}] &= (a_0 - i a_0 \mathbf{n} \vec{\sigma}) (b_0 + b_0 \frac{(\mathbf{m} - \mathbf{n} \times \mathbf{m})}{1 + n^2} \vec{\sigma}), \\ a_0^2 + a_0^2 n^2 &= 1, \quad \implies a_0 = \pm \frac{1}{\sqrt{1 + n^2}}, \\ b_0^2 - b_0^2 \frac{(\mathbf{m} - \mathbf{n} \times \mathbf{m})^2}{(1 + n^2)^2} &= 1, \quad \implies b_0 = \sqrt{1 + n^2}. \end{aligned} \quad (57)$$

In the same manner, one solves the problem with opposite order:

$$\begin{aligned} \pm [I - i(\mathbf{n} + i \mathbf{m}) \vec{\sigma}] &= (b_0 + \mathbf{b} \vec{\sigma}) (a_0 - i \mathbf{a} \vec{\sigma}), \\ &= a_0 b_0 + a_0 \mathbf{b} \vec{\sigma} - i b_0 \mathbf{a} \vec{\sigma} - i [\mathbf{a} \mathbf{b} - i (\mathbf{a} \times \mathbf{b}) \vec{\sigma}], \end{aligned} \quad (58)$$

which results in

$$\begin{aligned}
\pm [I - i(\mathbf{n} + i\mathbf{m}) \vec{\sigma}] &= (a_0 - i a_0 \mathbf{n} \vec{\sigma}) (b_0 + b_0 \frac{(\mathbf{m} + \mathbf{n} \times \mathbf{m})}{1 + n^2} \vec{\sigma}), \\
a_0^2 + a_0^2 n^2 &= 1, \implies a_0 = \pm \frac{1}{\sqrt{1 + n^2}}, \\
b_0^2 - b_0^2 \frac{(\mathbf{m} + \mathbf{n} \times \mathbf{m})^2}{(1 + n^2)^2} &= 1, \implies b_0 = \sqrt{1 + n^2}.
\end{aligned} \tag{59}$$

The $z = \lambda e^{i\sigma}$ – freedom in \mathbf{k} plays essential role in the factorizations:

$$\begin{aligned}
\pm [I + (-i\mathbf{n}' + \mathbf{m}') \vec{\sigma}] &= (a'_0 - i a'_0 \mathbf{n}' \vec{\sigma}) (b'_0 + b'_0 \frac{(\mathbf{m}' - \mathbf{n}' \times \mathbf{m}')}{1 + n'^2} \vec{\sigma}), \\
a'_0 &= \pm \frac{1}{\sqrt{1 + \lambda^2 n^2}}, \quad b'_0 = \frac{1}{\sqrt{1 + \lambda^2 n^2}}, \quad \mathbf{n}' \times \mathbf{m}' = \lambda^2 \mathbf{n} \times \mathbf{m};
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\pm [I + (-i\mathbf{n}' + \mathbf{m}')' \vec{\sigma}] &= (a'_0 - i a'_0 \mathbf{n}' \vec{\sigma}) (b'_0 + b'_0 \frac{(\mathbf{m}' + \mathbf{n}' \times \mathbf{m}')}{1 + n'^2} \vec{\sigma}), \\
a'_0 &= \pm \frac{1}{\sqrt{1 + \lambda^2 n^2}}, \quad b'_0 = \frac{1}{\sqrt{1 + \lambda^2 n^2}}, \quad \mathbf{n}' \times \mathbf{m}' = \lambda^2 \mathbf{n} \times \mathbf{m}.
\end{aligned} \tag{61}$$

7 Behavior of the non-linear constitutive relations under the Lorentz group

As noted above, in the frame of field theory in non-commutative space-time, extended electrodynamic equations minimally modified by the first order terms of non-commutativity θ_{ab} were constructed – those are usual Maxwell equations with special non-linear constitutive equations)

$$\begin{aligned}
\frac{\mathbf{D}}{\epsilon_0} &= \mathbf{E} + [(\mathbf{nE}) - (\mathbf{m}c\mathbf{B})] \mathbf{E} + [(\mathbf{mE}) + (\mathbf{n}c\mathbf{B})] c\mathbf{B} + (\mathbf{E}c\mathbf{B}) \mathbf{m} + \frac{1}{2}(\mathbf{E}^2 - c^2\mathbf{B}^2) \mathbf{n}, \\
\frac{\mathbf{H}}{c\epsilon_0} &= c\mathbf{B} + [(\mathbf{nE}) - (\mathbf{m}c\mathbf{B})] c\mathbf{B} - [(\mathbf{mE}) + (\mathbf{n}c\mathbf{B})] \mathbf{E} - (\mathbf{E}c\mathbf{B}) \mathbf{n} + \frac{1}{2}(\mathbf{E}^2 - c^2\mathbf{B}^2) \mathbf{m},
\end{aligned} \tag{62}$$

and inverse relations

$$\begin{aligned}
\mathbf{E} &= \mathbf{D}/\epsilon_0 + [\mathbf{mH}/c\epsilon_0 - \mathbf{nD}/\epsilon_0] \mathbf{D}/\epsilon_0 - [\mathbf{mD}/\epsilon_0 + \mathbf{nH}/c\epsilon_0] \mathbf{H}/c\epsilon_0 - \\
&\quad - (\mathbf{DH}c/\epsilon_0^2) \mathbf{m} + \frac{1}{2}(\mathbf{H}^2/c^2\epsilon_0^2 - \mathbf{D}^2/\epsilon_0^2) \mathbf{n}, \\
c\mathbf{B} &= \mathbf{H}/c\epsilon_0 + [\mathbf{mH}/c\epsilon_0 - \mathbf{nD}/\epsilon_0] \mathbf{H}/c\epsilon_0 + [\mathbf{mD}/\epsilon_0 + \mathbf{nH}/c\epsilon_0] \mathbf{D}/\epsilon_0 \\
&\quad + (\mathbf{DH}c/\epsilon_0^2) \mathbf{n} + \frac{1}{2}(\mathbf{H}^2/c^2\epsilon_0^2 - \mathbf{D}^2/\epsilon_0^2) \mathbf{m}.
\end{aligned} \tag{63}$$

In the used system SI, the dimensions of the quantities involved obey relations:

$$[E] = [\frac{D}{\epsilon_0}] = [cB] = \frac{H}{c\epsilon_0} = [\frac{1}{n}] = [\frac{1}{m}] .$$

Let us translate these formulas to Riemann-Silberstein-Majorana-Oppenheimer basis (more details and references see in [6]). Correspondingly, in the variables with simple transformation properties under the complex orthogonal group $SO(3.C)$, isomorphic to the Lorentz group L_+^\uparrow . To this end, it suffices to use the following variables

$$\mathbf{f} = \mathbf{E} + ic\mathbf{B} , \quad \mathbf{h} = \frac{1}{\epsilon_0}(\mathbf{D} + i\mathbf{H}/c) , \quad \mathbf{n} + i\mathbf{m} = \mathbf{K} ; \quad (64)$$

the constitutive equations read

$$\mathbf{h} = [1 + (\mathbf{f}^* \mathbf{K}^*)] \mathbf{f} + \frac{(\mathbf{f}^* \mathbf{f}^*)}{2} \mathbf{K} , \quad \mathbf{f} = [1 - (\mathbf{h}^* \mathbf{K}^*)] \mathbf{h} - \frac{\mathbf{h}^* \mathbf{h}^*}{2} \mathbf{K} . \quad (65)$$

These relations are inverse to each other within the accuracy of the first order terms in \mathbf{K} . With respect to the Lorentz group the constitutive equations behave themselves as follows:

$$\begin{aligned} \mathbf{h}' &= O\mathbf{h} , \quad \mathbf{f}' = O\mathbf{f} , \quad \mathbf{f}'^* = O^* \mathbf{f}^* , \quad \mathbf{K}' = O\mathbf{K} , \quad \mathbf{K}'^* = O^* \mathbf{K}^* , \\ \mathbf{h}' &= O[1 + ((O^*)^{-1} \mathbf{f}'^* (O^*)^{-1} \mathbf{K}'^*)] O^{-1} \mathbf{f}' + O \frac{(O^*)^{-1} \mathbf{f}'^* (O^*)^{-1} \mathbf{f}'^*}{2} O^{-1} \mathbf{K}' , \\ \mathbf{f}' &= O[1 - ((O^*)^{-1} \mathbf{h}'^* (O^*)^{-1} \mathbf{K}'^*)] O^{-1} \mathbf{h}' - O \frac{(O^*)^{-1} \mathbf{h}'^* (O^*)^{-1} \mathbf{h}'^*}{2} O^{-1} \mathbf{K}' . \end{aligned}$$

Allowing for orthogonality property of the elements of $SO(3.C)$ we arrive at

$$\mathbf{h}' = [1 + (\mathbf{f}'^* \mathbf{K}'^*)] \mathbf{f}' + \frac{\mathbf{f}'^* \mathbf{f}'^*}{2} \mathbf{K}' , \quad \mathbf{f}' = [1 - (\mathbf{h}'^* \mathbf{K}'^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}'^*}{2} \mathbf{K}' . \quad (66)$$

This means that the constitutive relations are explicitly covariant under the complex orthogonal group $SO(3.C)$. Evidently, above described 2-parametric small subgroups in $SO(3.C)$ leaving invariant non-commutativity parameters, complex 3-vectors \mathbf{K} and \mathbf{K}^* , provide us with subgroup in the Lorentz group, leaving invariant the nonlinear constitutive equations:

non-isotropic case

$$\begin{aligned} \mathbf{K} &= \mathbf{n} + i\mathbf{m} = K \mathbf{k} , \quad \Delta^2 = 1 , \quad \Delta = \mathbf{N} + i\mathbf{M} , \\ \mathbf{K}' &= O(\gamma, \Delta) \mathbf{K} = \mathbf{K} , \quad \mathbf{K}'^* = O^*(\gamma, \Delta) \mathbf{K}^* = \mathbf{K}^* , \\ &\text{subgroup } O(\gamma, \Delta) , \quad \gamma'' = \gamma' + \gamma , \\ \mathbf{h}' &= [1 + (\mathbf{f}'^* \mathbf{K}^*)] \mathbf{f}' + \frac{\mathbf{f}'^* \mathbf{f}'^*}{2} \mathbf{K} , \\ \mathbf{f}' &= [1 - (\mathbf{h}'^* \mathbf{K}^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}'^*}{2} \mathbf{K} , \end{aligned} \quad (67)$$

isotropic case

$$\begin{aligned}
\mathbf{K} &= \mathbf{n} + i\mathbf{m} = \boldsymbol{\Delta} , & \boldsymbol{\Delta}^2 &= 0 , \\
\mathbf{K}' &= O(z\boldsymbol{\Delta}) \mathbf{K} = \mathbf{K} , & \mathbf{K}'^* &= O^*(\gamma, \boldsymbol{\Delta}) \mathbf{K}^* = \mathbf{K}^* , \\
&\text{subgroup } O(z \boldsymbol{\Delta}) , & z'' &= z' + z , \\
\mathbf{h}' &= [1 + (\mathbf{f}'^* \mathbf{K}^*)] \mathbf{f}' + \frac{\mathbf{f}'^* \mathbf{f}^*}{2} \mathbf{K}' , \\
\mathbf{f}' &= [1 - (\mathbf{h}'^* \mathbf{K}^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}^*}{2} \mathbf{K} .
\end{aligned} \tag{68}$$

8 On constitutive relations and discrete dual symmetry

In absence of sources, Maxwell equations in media

$$\begin{aligned}
\operatorname{div} \mathbf{B} &= 0 , & \operatorname{rot} \mathbf{E} &= -\frac{\partial c\mathbf{B}}{\partial ct} , \\
\operatorname{div} \mathbf{D} &= 0 , & \operatorname{rot} \frac{\mathbf{H}}{c} &= \frac{\partial \mathbf{D}}{\partial ct}
\end{aligned} \tag{69}$$

can be combined into complex ones

$$\begin{aligned}
\operatorname{div} \left(\frac{\mathbf{D}}{\epsilon_0} + i c\mathbf{B} \right) &= 0 , \\
-i\partial_0 \left(\frac{\mathbf{D}}{\epsilon_0} + i c\mathbf{B} \right) + \operatorname{rot} \left(\mathbf{E} + i \frac{\mathbf{H}}{c} \right) &= 0 .
\end{aligned} \tag{70}$$

Variables with simple transformation properties under $SO(3.C)$ are

$$\mathbf{f} = \mathbf{E} + i c\mathbf{B} , \quad \mathbf{h} = \frac{1}{\epsilon_0} (\mathbf{D} + i\mathbf{H}/c) .$$

Eqs. (70) may be translated into

$$\begin{aligned}
\operatorname{div} \left(\frac{\mathbf{h} + \mathbf{h}^*}{2} + \frac{\mathbf{f} - \mathbf{f}^*}{2} \right) &= 0 , \\
-i\partial_0 \left(\frac{\mathbf{h} + \mathbf{h}^*}{2} + \frac{\mathbf{f} - \mathbf{f}^*}{2} \right) + \operatorname{rot} \left(\frac{\mathbf{f} + \mathbf{f}^*}{2} + \frac{\mathbf{h} - \mathbf{h}^*}{2} \right) &= 0 .
\end{aligned} \tag{71}$$

It has sense to introduce new variables

$$\mathbf{G} = \frac{\mathbf{h} + \mathbf{f}}{2} , \quad \mathbf{R} = \frac{\mathbf{h}^* - \mathbf{f}^*}{2} , \tag{72}$$

they are vectors of different type under $SO(3.C)$ group: $\mathbf{G}' = O \mathbf{G}$, $\mathbf{R}' = O^* \mathbf{R}$. Accordingly, Maxwell equations read

$$\begin{aligned}
\operatorname{div} \mathbf{G} + \operatorname{div} \mathbf{R} &= 0 , \\
-i\partial_0 \mathbf{G} + \operatorname{rot} \mathbf{G} - i\partial_0 \mathbf{R} - \operatorname{rot} \mathbf{R} &= 0 ,
\end{aligned} \tag{73}$$

these are invariant under dual rotations: $e^{i\chi} \mathbf{G} = \mathbf{G}'$, $e^{i\chi} \mathbf{R} = \mathbf{R}'$, which can be translated to variables \mathbf{h}, \mathbf{f} :

$$\mathbf{h}' = \cos \chi \mathbf{h} + i \sin \chi \mathbf{f}, \quad \mathbf{f}' = i \sin \chi \mathbf{h} + \cos \chi \mathbf{f}.$$

Following to [4], the dual rotations for \mathbf{K} is taken in the form $\mathbf{K}' = e^{i\chi} \mathbf{K}$. Let us consider behavior of the constitutive relations with respect to the dual rotation (for brevity, let $\epsilon_0 = 1, c = 1$):

$$\mathbf{h} = [1 + (\mathbf{f}^* \mathbf{K}^*)] \mathbf{f} + \frac{(\mathbf{f}^* \mathbf{f}^*)}{2} \mathbf{K}, \quad \mathbf{f} = [1 - (\mathbf{h}^* \mathbf{K}^*)] \mathbf{h} - \frac{\mathbf{h}^* \mathbf{h}^*}{2} \mathbf{K}.$$

We immediately note three discrete operations leaving invariant the constitutive relations:

$$\begin{aligned} (1) \quad & \chi = \frac{\pi}{2}, \quad \mathbf{h}' = i \mathbf{f}, \quad \mathbf{f}' = i \mathbf{h}, \quad \mathbf{K}' = i \mathbf{K}, \\ & \mathbf{f}' = [1 - (\mathbf{h}'^* \mathbf{K}'^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}'^*}{2} \mathbf{K}', \quad \mathbf{h}' = [1 + (\mathbf{f}'^* \mathbf{K}'^*)] \mathbf{f}' + \frac{(\mathbf{f}'^* \mathbf{f}'^*)}{2} \mathbf{K}', \\ (2) \quad & \chi = \pi, \quad \mathbf{f}' = -\mathbf{f}, \quad \mathbf{h}' = -\mathbf{h}, \quad \mathbf{K}' = -\mathbf{K}, \\ & \mathbf{h}' = [1 + (\mathbf{f}'^* \mathbf{K}'^*)] \mathbf{f}' + \frac{(\mathbf{f}'^* \mathbf{f}'^*)}{2} \mathbf{K}', \quad \mathbf{f}' = [1 - (\mathbf{h}'^* \mathbf{K}'^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}'^*}{2} \mathbf{K}', \\ (3) \quad & \chi = \frac{3\pi}{2}, \quad \mathbf{h}' = -i \mathbf{f}, \quad \mathbf{f}' = -i \mathbf{h}, \quad \mathbf{K}' = -i \mathbf{K}, \\ & \mathbf{f}' = [1 - (\mathbf{h}'^* \mathbf{K}'^*)] \mathbf{h}' - \frac{\mathbf{h}'^* \mathbf{h}'^*}{2} \mathbf{K}', \quad \mathbf{h}' = [1 + (\mathbf{f}'^* \mathbf{K}'^*)] \mathbf{f}' + \frac{(\mathbf{f}'^* \mathbf{f}'^*)}{2} \mathbf{K}'. \end{aligned} \tag{74}$$

Together with the unit transform

$$(4) \quad \chi = 0, \quad \mathbf{f}' = +\mathbf{f}, \quad \mathbf{h}' = +\mathbf{h}, \quad \mathbf{K}' = +\mathbf{K}$$

we have the discrete group of four element with simple structure: $\{1, -1, +i, -i\}$.

Let us consider action of continuous dual rotations on constitutive equations. It is convenient to use the variables \mathbf{G}, \mathbf{R} :

$$\begin{aligned} \mathbf{G} + \mathbf{R}^* &= \mathbf{h}, & \mathbf{G}^* + \mathbf{R} &= \mathbf{h}^*, \\ \mathbf{G} - \mathbf{R}^* &= \mathbf{f}, & \mathbf{G}^* - \mathbf{R} &= \mathbf{f}^*, \end{aligned} \tag{75}$$

then

$$\begin{aligned} \mathbf{G} + \mathbf{R}^* &= [1 + (\mathbf{G}^* - \mathbf{R}) \mathbf{K}^*] (\mathbf{G} - \mathbf{R}^*) + \frac{(\mathbf{G}^* - \mathbf{R})(\mathbf{G}^* - \mathbf{R})}{2} \mathbf{K}, \\ \mathbf{G} - \mathbf{R}^* &= [1 - (\mathbf{G}^* + \mathbf{R}) \mathbf{K}^*] (\mathbf{G} + \mathbf{R}^*) - \frac{(\mathbf{G}^* + \mathbf{R})(\mathbf{G}^* + \mathbf{R})}{2} \mathbf{K}, \end{aligned}$$

from whence it follows

$$\begin{aligned} 2(\mathbf{G}^* \mathbf{R}) \mathbf{K} + (\mathbf{G}^* \mathbf{K}^*) \mathbf{R}^* + (\mathbf{R} \mathbf{K}^*) \mathbf{G} &= 0, \\ 2\mathbf{R}^* &= (\mathbf{G}^* \mathbf{K}^*) \mathbf{G} + (\mathbf{R} \mathbf{K}^*) \mathbf{R}^* + \frac{1}{2} (\mathbf{G}^* \mathbf{G}^* + \mathbf{R} \mathbf{R}) \mathbf{K}. \end{aligned} \quad (76)$$

When $\mathbf{K} = 0$, eq. (76) gives $0 = 0$, $\mathbf{R} = 0$, $\implies \mathbf{h} = \mathbf{f}$, which coincides with constitutive relations in vacuum. With respect to the dual rotation

$$\begin{aligned} e^{i\chi} \mathbf{G} &= \mathbf{G}', \quad e^{-i\chi} \mathbf{G}^* = \mathbf{G}'^*, \quad e^{i\chi} \mathbf{R} = \mathbf{R}', \\ e^{-i\chi} \mathbf{R}^* &= \mathbf{R}'^*, \quad e^{i\chi} \mathbf{K} = \mathbf{K}', \quad e^{-i\chi} \mathbf{K}^* = \mathbf{K}'^* \end{aligned}$$

eqs. (76) transform into

$$\begin{aligned} &2 (e^{i\chi} \mathbf{G}'^* e^{-i\chi} \mathbf{R}') e^{-i\chi} \mathbf{K}' \\ &+ (e^{i\chi} \mathbf{G}'^* e^{i\chi} \mathbf{K}'^*) e^{i\chi} \mathbf{R}'^* + (e^{-i\chi} \mathbf{R}' e^{i\chi} \mathbf{K}'^*) e^{-i\chi} \mathbf{G}' = 0, \\ 2 e^{i\chi} \mathbf{R}'^* &= (e^{i\chi} \mathbf{G}'^* e^{i\chi} \mathbf{K}'^*) e^{-i\chi} \mathbf{G}' + (e^{-i\chi} \mathbf{R}' e^{i\chi} \mathbf{K}'^*) e^{i\chi} \mathbf{R}'^* \\ &+ \frac{1}{2} [e^{i\chi} \mathbf{G}'^* e^{i\chi} \mathbf{G}'^* + e^{-i\chi} \mathbf{R}' e^{-i\chi} \mathbf{R}'] e^{-i\chi} \mathbf{K}'. \end{aligned}$$

Requiring invariance of these equations we arrive at two simple equations with evident solution

$$\begin{aligned} e^{-i\chi} &= e^{+3i\chi}, \quad e^{+i\chi} = e^{-3i\chi}, \quad e^{+4i\chi} = 1, \\ e^{i\chi} &= 1, -1, +i, -i. \end{aligned} \quad (77)$$

Therefore, only discrete dual transformation leaves invariant the non-linear constitutive equations, it corresponds to $e^{i\chi} = \pm i$. Thus, the dual symmetry status in non-commutative electrodynamics differs with that in ordinary linear Maxwell theory in commutative space, this fact is to be interpreted in physical terms.

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